20/10/2021

<u>Obs:-</u> Exact-Clique $\in \Sigma_2^P$

<u>Proof sketch:-</u>

input : $\langle G, K \rangle$

$V(\underbrace{\langle G, k\rangle}_{x:=}, u_1, u_2)$ set of vertices

set of vertices that forms the largest Clique

→ if $x \neq \langle G, k\rangle$ → REJECT

→ if u_1 doesn't encode a clique of size k → REJECT

→ if u_2 encodes a clique of size at least $k+1$ → REJECT

→ O/W ACCEPT.

$\exists u_1 \quad \forall u_2 \quad V(x, u_1, u_2) = 1.$

$x = \langle G, k\rangle \in$ Exact-Clique

then $\exists u_1 \quad \forall u_2 \quad V(x, u_1, u_2) = 1.$

$x = \langle G, k\rangle \notin$ Exact-Clique

then (i) largest Clique $> k$ → $\forall u_1 \ \exists u_2$

s.t. $V(x, u_1, u_2) = 0$

(ii) largest Clique $< K \rightarrow \forall u_1 \; \forall u_2$

$$V(x, u_1, u_2) = 0$$

<u>Defn</u>: (Π_2^P) : $L \in \Pi_2^P$ if $\exists$ a poly-time DTM V and poly $p(\cdot)$ s.t. $\forall x \in \{0,1\}^*$

$$x \in L \Leftrightarrow \forall u_1 \in \{0,1\}^{p(|x|)} \; \exists u_2 \in \{0,1\}^{p(|x|)}$$

s.t. $V(x, u_1, u_2) = 1$.

<u>Q</u>: Min-Ckt-Size $\in \Pi_2^P$

$\{\langle C\rangle \mid C$ is the smallest ckt computing the function represented by $C\}$

<u>Sketch</u>: $\forall C'$ s.t. $\text{size}(C') < \text{size}(C)$

$\exists x \in \{0,1\}^n$ s.t. $C'(x) \neq C(x)$.

Defn:- Σ_i^P ; $i \in \mathbb{N}$

$L \in \Sigma_i^P$ if $\exists$ a poly-time DTM V and polynomial $p(\cdot)$ s.t. $\forall x \in \{0,1\}^*$

$$x \in L \Leftrightarrow \exists u_1 \in \{0,1\}^{p(|x|)} \; \forall u_2 \in \{0,1\}^{p(|x|)} \; \exists u_3 \in \{0,1\}^{p(|x|)}$$

$$\cdots Q_i u_i \in \{0,1\}^{p(|x|)} \text{ s.t. } V(x, u_1, \ldots, u_i) = 1$$

where $Q_i \in \{\forall, \exists\}$ is $\forall$ (or $\exists$) if i is even (or odd).

Defn: Π_i^P : $x \in L \Leftrightarrow \forall \exists \forall \cdots V(x, u_1, \ldots, u_i) = 1$

Obs:- $\Sigma_i^P \subseteq \Sigma_{i+1}^P$

Proof sketch: $\Sigma_2^P \subseteq \Sigma_3^P := \exists \forall \underline{\exists} P$

Σ_2^P !! $\exists \forall \underline{P}$

Obs:- $\Pi_i^P \subseteq \Pi_{i+1}^P$

$\Pi_2^P := \forall \exists \underline{P}$; $\Pi_3^P := \forall \exists \underline{\forall} P$

dummy variables

$(x_1 \wedge x_2) \vee x_3$

$x_1 = 0$
$x_2 = 1$
$x_3 = 1$

$\exists y\ (x_1 \wedge x_2) \vee x_3$

$x_1 = 0$
$x_2 = 1$
$x_3 = 1$

$\forall y\ (x_1 \wedge x_2) \vee x_3$

$x_1 = 0$
$x_2 = 1$
$x_3 = 1$

$L \in \Pi_2^P$

$x \in L \Leftrightarrow \forall u_1\ \exists u_2\ \underline{V(x, u_1, u_2)} = 1$

$x \in L \Leftrightarrow \forall u_1\ \exists u_2\ \forall u_3\ V(x, u_1, u_2, u_3) = 1$

$\Updownarrow$

$V(x, u_1, u_2) = 1$

Obs:- $\Sigma_2^P \subseteq \Pi_3^P$?

$$\Sigma_2^P = \exists \forall P \qquad \Pi_3^P = \underline{\forall} \exists \forall P$$

dummy variables.

Obs:- Σ_i^P — Σ_{i+1}^P, Π_{i+1}^P

Obs:- $\Sigma_0^P = P = \Pi_0^P$

$$
\begin{tikzcd}
 & PH & \\
 & \vdots & \\
\Sigma_3^P & & \Pi_3^P \\
\Sigma_2^P \arrow[u] \arrow[urr] & & \Pi_2^P \arrow[u] \arrow[ull] \\
NP = \Sigma_1^P \arrow[u] \arrow[urr] & & \Pi_1^P = \text{co-}NP \arrow[u] \arrow[ull] \\
 & P = \Sigma_0^P = \Pi_0^P \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Defn:- Polynomial Hierarchy (PH)

$$PH := \bigcup_{i \geq 0} \Sigma_i^P = \bigcup_{i \geq 0} \Pi_i^P$$

Obs:- $PH \subseteq PSPACE$

TQBF is PSPACE-Complete

$$L \subseteq NL \subseteq P \subseteq NP \subseteq PH \subseteq PSPACE \subseteq EXP$$

Q:- what happens if $P = NP$?

$P = NP \Rightarrow NP = coNP$

$P = NP = coNP$

$P = NP$ intuitively $NP = \exists P$

$\Rightarrow \exists P = P$

$P = coNP \Rightarrow \forall P = P$

$\quad\;\; \|$

$\quad\; \forall P$

$\Sigma_2^P = \exists \forall P \xrightarrow{\forall P = P} \exists P$

$\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \exists P = P$

$\qquad\qquad\qquad\qquad\qquad\qquad P$

$\Pi_2^P = \forall \exists P = \forall P = P$

Lemma:- if $P = NP$ then $PH = P$

Proof Sketch:- by induction on i

let $L \in PH$.

$\Rightarrow \exists\, i$ s.t. $L \in \Sigma_i^P$ (or Π_i^P)

$L \in \Sigma_i^P \Rightarrow \exists$ poly-time DTM V and poly $p(\cdot)$ s.t.

$x \in L \Leftrightarrow \exists u_1 \in \{0,1\}^{p(|x|)} \forall u_2 \in \{0,1\}^{p(|x|)} \cdots Q_i u_i$

$$V(x, u_1, \ldots, u_i) = 1.$$

Define $L' := \{ \langle x, u_1 \rangle \mid \forall u_2 \exists u_3 \cdots Q_i u_i \; V(x, u_1, u_2, \ldots, u_i) = 1 \}$

$(x, u_1) \in L'$ iff $\forall u_2 \exists u_3 \cdots Q_i u_i$

$$V(x, u_1, u_2, \ldots, u_i) = 1.$$

$$L' \in \Pi^P_{i-1}$$

$$\boxed{\begin{array}{l} \underline{\text{Base Case:}} \quad i = 1, \quad \Sigma_1^P = NP, \text{ by assumption.} \\ \qquad\qquad\qquad\qquad\qquad\qquad \Pi_1^P = coNP \text{ by assumption.} \end{array}}$$

Induction Hypothesis implies $\Pi^P_{i-1} = P$

$\Rightarrow L' \in P$

$\Rightarrow \exists$ a poly-time DTM $M(x, u_1) = 1$

if $(x, u_1) \in L'$

$x \in L \iff \exists u_1$ s.t. $(x, u_1) \in L'$

$$x \in L \iff \underbrace{\exists u_1 \text{ s.t. } M(x, u_1) = 1}_{NP}$$

$\Rightarrow L \in NP$

by our assumption $NP = P$

$\Rightarrow L \in P. \Rightarrow \Sigma_i^P \subseteq P$

$\blacksquare$

Thm :- if $\Sigma_i^P = \Pi_i^P$ then $PH = \Sigma_i^P = \Pi_i^P$

In words, PH collapses to level i.

Complete Problems: usual Karp-reduction

Poly-time many one reduction

L is Σ_i^P-complete if $L \in \Sigma_i^P$ and

$\forall L' \in \Sigma_i^P \; L' \leq_p L$.

Similarly for Π_i^P-completeness.

Σ_i^P-complete language:

$\exists x_1 \in \{0,1\}^p \; \forall x_2 \in \{0,1\}^p \; \exists x_3 \ldots Q_i x_i$

s.t $\varphi(x_1, x_2, \ldots x_i) = 1$.

Π_i^p - Complete language :

$\forall x_1 \exists x_2 \dots Q_i x_i$ s.t. $\varphi(x_1, \dots, x_i) = 1$.

Q :- what language is complete for the class PH ? NO..

Suppose $\exists L$ complete for PH unless PH collapses to some level i.

$\Rightarrow L \in \Sigma_i^p$ (or Π_i^p)

and every language L' in PH poly-time reduces to L

$L' \leq_p L$ and $L \in \Sigma_i^p$

$\Rightarrow L' \in \Sigma_i^p$

$\Rightarrow PH \subseteq \Sigma_i^p$

$\Rightarrow \Pi_i^p \subseteq \Sigma_i^p \quad (\Sigma_i^p = \Pi_i^p)$

$\Rightarrow$ PH collapses to level i.

Obs:- $\Pi_i^P = \text{co}-\Sigma_i^P := \{ L \mid \bar{L} \in \Sigma_i^P \}$

$P \subseteq NP \cap \text{co-}NP$

$NP \cap \text{co-}NP \subseteq P$?

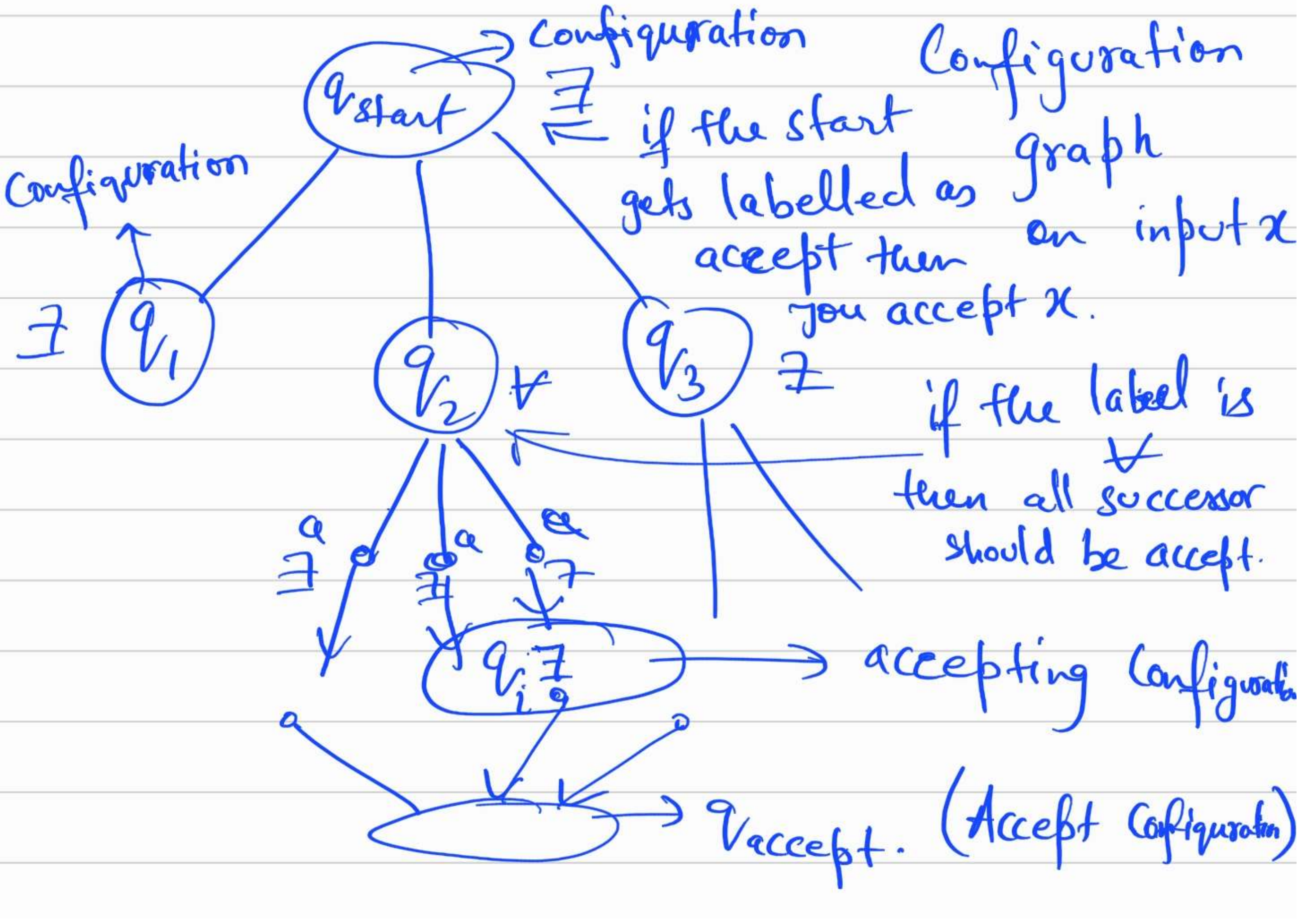
Alternating Turing Machines.

generalization NDTM

set of states. $:= \{ q_1, \ldots, q_t \}$

every state has a label in $\{ \exists, \forall \}$

$q_i \to \exists \;, \quad q_j \to \forall$



alternating time machine M, input x.

M accepts x if

consider $G_{M,x}$ (configuration graph on input x).

Label the vertices (configurations) as follows:

- C_{accept} $\longrightarrow$ "ACCEPT"

- if C is labelled with $\exists$ and one of its successor is labelled "ACCEPT" then label C as "ACCEPT"

- if C is labelled $\forall$ and all of its successor (out-neighbor) are labeled "ACCEPT" then label C as "ACCEPT"

- if C_{start} gets labelled "ACCEPT" in this manner then accept the input x.