Space Complexity

- $\text{PSPACE} := \bigcup_{c \geq 0} \text{DSPACE}(n^c)$
- $\text{NPSPACE} := \bigcup_{c \geq 0} \text{NSPACE}(n^c)$
- $L = \text{LOGSPACE} := \text{DSPACE}(\log n)$
- $\text{NL} := \text{NSPACE}(\log n)$

Lemma :- $L \subseteq NL \subseteq P \subseteq \text{PSPACE} \subseteq \text{NPSPACE}$

Lemma :- $\text{SPACE}(f(n)) \subseteq \text{NSPACE}(f(n)) \subseteq \text{DTIME}(2^{O(f(n))})$

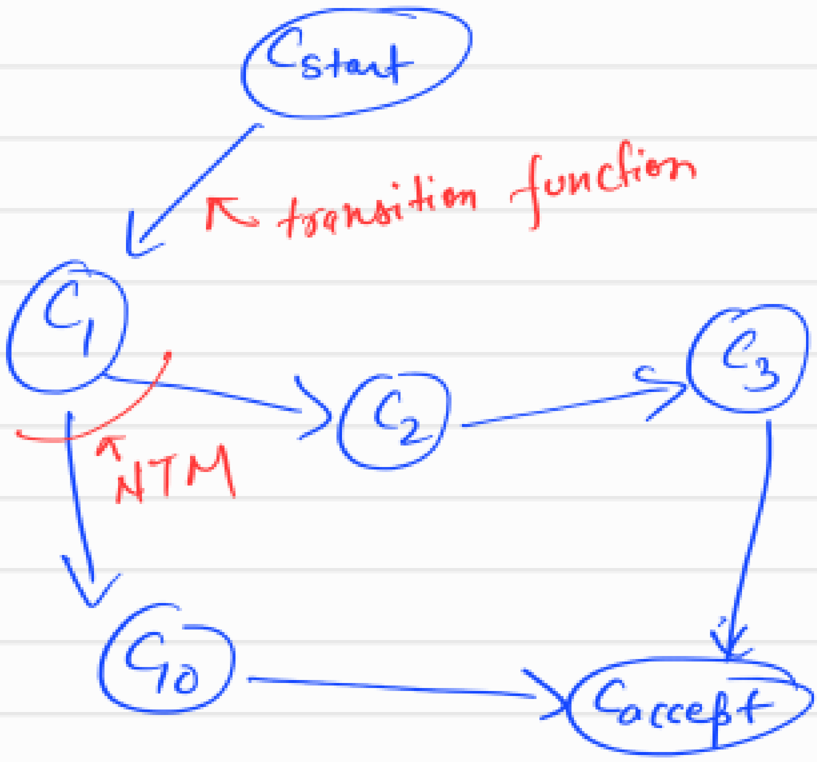
Configuration graphs ?

TM M, runs in space $f(n)$ Configurations encode
- input
- input symbol being read
- locations of the tape head
- Contents of the work tape

bits that is needed to encode a configuration $= c \cdot f(n) = O(f(n))$

WLOG, unique accepting configuration.

Configuration graph : space $f(n)$



if DTM, out-degree of every node is $= 1$.

$\#$ Configurations := $\#$ vertices in configuration graph $= 2^{c \cdot f(n)}$

$= 2^{O(f(n))}$

M accepts a string x

iff $\exists$ a path in $G_{M,x}$ from C_{start} to C_{accept}.

→ PSPACE / NPSPACE

reduction :- Karp-reduction ; poly-time reduction

$\leq_p$

Defn:- $L \subseteq \{0,1\}^*$ is PSPACE-hard

if $\forall L' \in$ PSPACE, $L' \leq_p L$.

Furthermore, $L \in$ PSPACE then

L is PSPACE-complete.

Defn:- STMSAT $:= \{ \langle M, x, 1^n \rangle \mid$ DTM M accepts x in n space$\}$

$|x| = n$

STMSAT is PSPACE-complete.

M is run on $y \neq x$
then suppose it needs more than n space
then $\langle M, y, 1^n \rangle \notin$ STMSAT.

Defn:- QBF = Quantified Boolean Formulae.

$$Q_1 x_1 \, Q_2 x_2 \ldots Q_n x_n \underbrace{\varphi(x_1, x_2, \ldots, x_n)}$$

where $Q_i \in \{\exists, \forall\}$ $\qquad$ φ is Boolean formulae.

$\rightarrow$ $\underline{\exists x_1 \, \forall x_2 \, \exists x_3 \, \forall x_4 \, (x_1 \wedge x_2) \vee (x_3 \wedge x_4) \wedge (x_5 \vee x_6)}$

$(x_1 \wedge x_2) \vee (x_3 \wedge x_4) \wedge (x_5 \vee x_6)$ formula with free variables.

$\rightarrow \quad \exists x_1 \; \forall x_2 \quad (x_1 \wedge x_2) \vee (\neg x_1 \wedge \neg x_2)$

$\exists x_1 \in \{0,1\} \;\; \forall x_2 \in \{0,1\} \quad \underbrace{(x_1 \wedge x_2) \vee (\neg x_1 \wedge \neg x_2)}$

$\underline{\exists x_1 \; \forall x_2 \; [x_1 = x_2]}$ $\qquad x_1 = x_2$

False $\notin$ TQBF

$x_1 = 0, \quad x_2 = 0$

$x_1 = 0, \quad x_2 = 1$

$\underline{\forall x_1 \; \exists x_2 \; [x_1 = x_2]}$ = Statement $\in$ TQBF

TRUE

$x_1 = 0, \quad x_2 = 0$

$x_1 = 1, \quad x_2 = 1$

$\exists x_1 \, \exists x_2 \, [x_1 = x_2] \leftarrow$ encoding SAT as QBF

QBF vs BF with free vars.

$\forall x_1 \forall x_2 \forall x_3 \cdots \forall x_n \;\; \varphi(x_1, \ldots, x_n) \leftarrow$ TAUTOLOGY

TQBF := { True QBFs }

<u>Thm</u> :- TQBF is PSPACE-Complete.

TQBF $\in$ PSPACE.

QBF $\psi = Q_1 x_1\, Q_2 x_2 \cdots Q_n x_n\ \varphi(x_1, \ldots, x_n)$

$\text{size}(\varphi) := |\varphi| = m.$

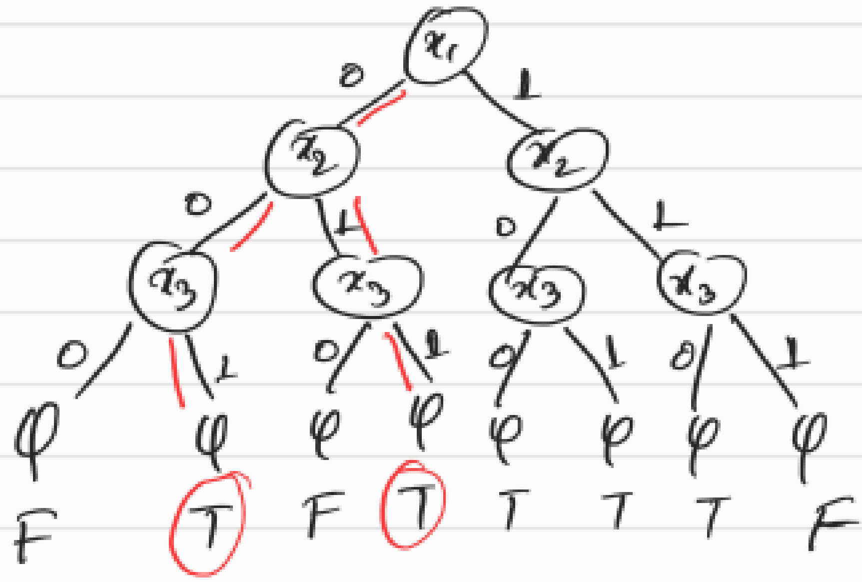
where $Q_i \in \{\exists, \forall\}$

$\triangleright$ take an assignment $z \in \{0,1\}^n$ to the variables $x_1, \ldots, x_n$.

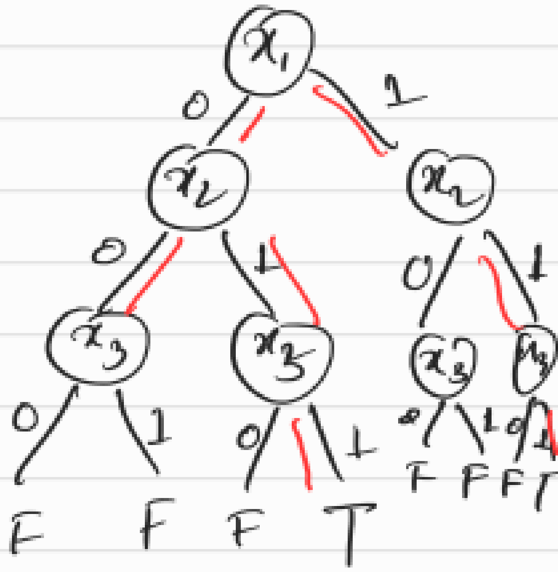
Evaluate $\varphi(z_1, \ldots, z_n)$. How much space?

Space $= O(m+n)$

$\psi' := \exists x_1\ \forall x_2\ \exists x_3\quad \varphi(x_1, x_2, x_3)$



$\psi' \in$ TQBF? ✓



$\varphi \in SAT$

$\psi' \notin$ TQBF.

SAT $= \exists x_1 \exists x_2 \cdots \exists x_n\ \varphi(x_1, \ldots, x_n)$

Pseudo-Algo: $(A(\psi))$

For i in 1 to n.

if $Q_i = \exists$ then $A(\psi|_{x_i=0}) \vee A(\psi|_{x_i=1})$

if $Q_i = \forall$ then $A(\psi|_{x=0}) \wedge A(\psi|_{x_i=1})$

$$S(n,m) \le S(n-1,m) + O(m)$$

$$S(n,m) = O(nm)$$

$\blacksquare$

TQBF is PSPACE-Hard.

$L \in$ PSPACE $\quad L \le_p$ TQBF.

<u>CHESS</u> :- Does player with Black pieces has a winning Strategy?

$\exists B \; \forall W \; \exists B \; \forall W \; \cdots\cdots \; \varphi(B, W, \ldots, W)$

encodes chess configuration, transition,

Fix M decides $L \subseteq \{0,1\}^*$ in space n^k

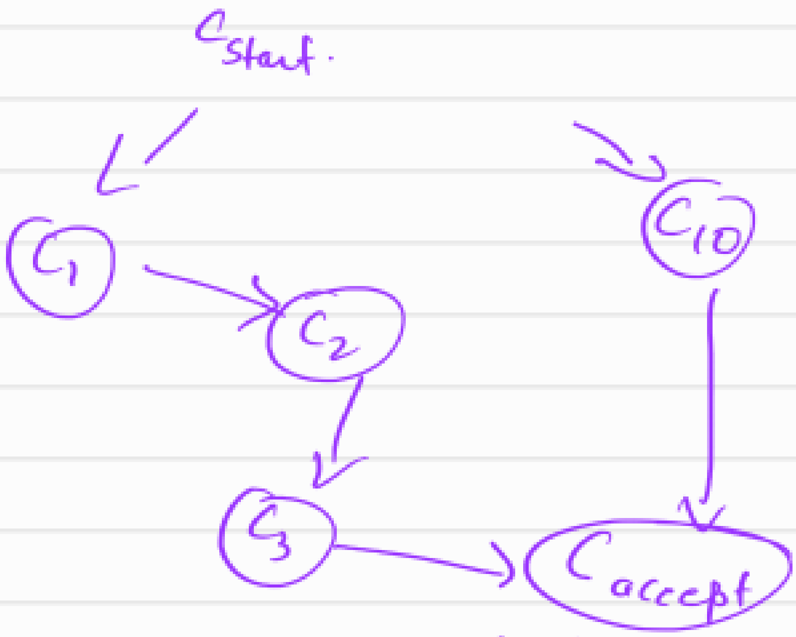
Given $x \in \{0,1\}^n$ where $k \in \mathbb{N}$.

need a poly time algo f s.t.

$$x \in L \iff f(x) \in \text{TQBF}$$

QBF

Configuration graph $G_{M,x}$, n



c_1

c_2

.

.

c_d

$$\# \text{ Configurations } = 2^{O(n^k)}$$

$\#$ bits needed to represent a configuration

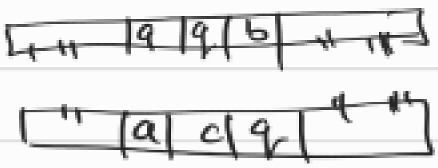
$$= O(n^k)$$

$x \in L \iff \exists$ a path from C_{start} to C_{accept} in $G_{M,x}$

$C_i , C_j \longrightarrow$ Can you go from $C_i \to C_j$ in one step?

C_i

C_j



$$\delta(q, b) \Rightarrow (q_1, c, R)$$

$$\varphi_{move}(c_i, c_j) = \text{True iff } C_i \to C_j \text{ in } G_{M,x}$$

$$|\varphi_{move}| \leq O(n^k)$$

Ans :- $\exists C_1 \exists C_2 \exists C_3 \cdots \exists C_\ell$ s.t.

$$\varphi_{move}(C_{start}, C_1) \wedge \varphi_{move}(C_1, C_2) \wedge \varphi_{move}(C_2, C_3)$$

$$\cdots \wedge \cdots \wedge \varphi_{move}(C_\ell, C_{accept})$$

$\ell = ?$ How many vertices can be in a path from C_{start} to C_{accept}? in $G_{M,x}$

all vertices $= 2^{O(n^k)}$

Second Attempt :- want to check that

$\exists$ a path of length $\leq 2^{O(n^k)}$ from C_{start} to C_{accept}.

$\exists C_{mid}$ [$\exists$ a path of length $\frac{1}{2} \cdot 2^{O(n^k)}$ from C_{start} to C_{mid}] $\wedge$ [$\exists$ a path of length $\frac{1}{2} \cdot 2^{O(n^k)}$ from C_{mid} to C_{accept}]

$\psi_\ell(C_i, C_j) :=$ $\exists$ path of length $\leq 2^\ell$ between C_i and C_j.

$\exists\, C_{mid} \quad \psi_{\ell-1}(c_i, c_{mid}) \wedge \psi_{\ell-1}(c_{mid}, c_j)$

$$\psi_0(c_i, c_j) = \underline{\varphi_{move}(c_i, c_j)} \vee \underline{[c_i = c_j]}$$

$\underline{\psi_{O(n^k)}(c_{start}, c_{accept})}$:= output of the reduction

$$|\psi_{O(n^k)}| \stackrel{?}{=} O(n^k)$$

$$|\psi_0| = O(n^k)$$

$$\psi_\ell(c_i, c_j) = \underline{\exists\, C_{mid}} \ \underline{\psi_{\ell-1}(c_i, c_{mid})} \wedge \underline{\psi_{\ell-1}(c_{mid}, c_j)}$$

$$|\psi_\ell| \le O(n^k) + 2 \cdot |\psi_{\ell-1}|$$

$$|\psi_{O(n^k)}| = O\left(2^{O(n^k)} \cdot n^k\right)$$

<u>Attempt 3</u> :- $\psi_\ell(c_i, c_j) =$

$\exists\, C_{mid} \ \forall D_1 \ \forall D_2$

$$\Big[(D_1, D_2) = \underline{(C_i, C_{mid})} \vee (D_1, D_2) = \underline{(C_{mid}, C_j)} \Big]$$

$F \Rightarrow T/F$

$F \Rightarrow F$; $F \Rightarrow T$

$$\Downarrow$$

$$\psi_{\ell-1}(D_1, D_2)$$

$$\left[B(\vec{0}, \ldots) \wedge B(1, \ldots) \right]$$

$$\text{III}$$

$$\forall x_1 \; B(x_1, \ldots)$$

$$\boxed{a \Rightarrow b \equiv \neg a \vee b}$$

$$|\psi_\ell| \leq O(n^k) + |\psi_{\ell-1}|$$

$$\Rightarrow |\psi_{O(n^k)}| \leq O(n^{2k})$$

$$\psi_{O(n^k)}(C_{start}, C_{accept}) \leftarrow \text{output QBF}$$

$$|\psi_{O(n^k)}| \leq O(n^{2k}) \leftarrow \text{poly.}$$

$$\Rightarrow x \in L \iff \psi_{O(n^k)} \in TQBF$$

$\blacksquare$

Comment:

1) Does it matter if M was non-det?

NO!

$\Rightarrow$ TQBF is NPSPACE-hard.

$\Rightarrow$ NPSPACE = PSPACE.

Also follows Savitch's thm.

$NSPACE(f(n)) \subseteq SPACE(f(n)^2)$

$\Rightarrow$ NPSPACE = coNPSPACE

$coNPSPACE = \{L \mid \bar{L} \in NPSPACE\}$

$L \in coNPSPACE \Rightarrow \bar{L} \in NPSPACE$

$\Rightarrow \bar{L} \in PSPACE$

$\Rightarrow L \in PSPACE$

$\blacksquare$

Don't expect

NP = coNP.

if NP = coNP $\Rightarrow$ P = NP.

$$\Psi_\ell \doteq \exists C_m \forall D_1 \forall D_2 \left[(D_1, D_2) = (C_i, C_m) \vee (C_m, C_j) \right] \Rightarrow \Psi_{\ell-1}(D_1, D_2)$$