Boolean function complexity

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In this lecture we study the Fourier transform of functions computable by constant depth circuits. We will see that such functions have a very *light* Fourier tail beyond level $(\log s)^d$.

More formally, we will prove the following structure theorem on the Fourier spectrum of a function computed by constant depth circuits of size s and depth d.

Theorem 1 (Linial-Mansour-Nisan'93). *Let $f\colon \{0,1\}^n \to \{0,1\}$ be a Boolean function computed by a circuit of size s and depth d. Further let $k > 0$. Then,*

$$\sum_{S \subseteq [n]\colon |S| > k} \widehat{f}(S)^2 \leq 2 \cdot s \cdot 2^{-k^{1/d}/10}.$$

Before we prove the theorem we need to understand how the Fourier transform of f relates to the Fourier transform of its restrictions.

Proposition 2. *Let $f : \{-1,1\}^n \to \mathbb{R}$, $S \subseteq [n]$, and $p \in [0,1]$. Then,*

$$\mathbb{E}_{\rho \in R_p}\left[\widehat{f|_\rho}(S)\right] = \widehat{f}(S) \cdot p^{|S|},$$

where R_p is the set of p-random restrictions.

Proof. Recall, in a p-random restriction, independently for each variable we leave it unset with prob. p and set it to 1 or -1 with equal prob. $(1-p)/2$. This random process can equivalently be thought as *first* choosing a random set T as follows: independently for each $i \in [n]$, include $i \in T$ with prob. p. And *then* sampling a $z \in \{-1,1\}^{|\overline{T}|}$ uniformly at random and assigning it to the variables in $[n] \setminus T$.

Therefore,

$$\begin{aligned}
\mathbb{E}_{\rho \in R_p}\left[\widehat{f|_\rho}(S)\right] &= \mathbb{E}_{T \subseteq_p [n]}\, \mathbb{E}_{z \in \{-1,1\}^{|\overline{T}|}}\left[\widehat{f|_{T,z}}(S)\right] \\
&= \mathbb{E}_{T \subseteq_p [n]}\, \mathbb{E}_{z \in \{-1,1\}^{|\overline{T}|}}\left[\mathbb{1}_{S \subseteq T} \cdot \left(\sum_{S' \subseteq \overline{T}} \widehat{f}(S \cup S') \prod_{j \in S'} z_j\right)\right] \\
&= \mathbb{E}_{T \subseteq_p [n]}\left[\mathbb{1}_{S \subseteq T} \cdot \left(\sum_{S' \subseteq \overline{T}} \widehat{f}(S \cup S')\, \mathbb{E}_{z \in \{-1,1\}^{|\overline{T}|}}\left[\prod_{j \in S'} z_j\right]\right)\right] \\
&= \mathbb{E}_{T \subseteq_p [n]}\left[\mathbb{1}_{S \subseteq T} \cdot \widehat{f}(S)\right] \\
&= \widehat{f}(S) \cdot p^{|S|}.
\end{aligned}$$

The fourth equality follows from the fact $\mathbb{E}[\prod_{j \in S'} z_j] = 0$ iff $S' \neq \emptyset$. $\square$

We now estimate the expected squared Fourier coefficients.

Proposition 3. *Let* $f : \{-1,1\}^n \to \mathbb{R}$*,* $S \subseteq [n]$*, and* $p \in [0,1]$*. Then,*

$$\mathbb{E}_{\rho \in R_p}\left[\widehat{f|_\rho}(S)^2\right] = \sum_{U \subseteq [n]} \widehat{f}(U)^2 \cdot \Pr_\rho[\rho^{-1}(*) \cap U = S],$$

where R_p *is the set of* p*-random restrictions.*

Proof. As before,

$$\begin{aligned}
\mathbb{E}_{\rho \in R_p}\left[\widehat{f|_\rho}(S)^2\right] &= \mathbb{E}_{T \subseteq_p [n]} \mathbb{E}_{z \in \{-1,1\}^{|\overline{T}|}} \left[\widehat{f|_{T,z}}(S)^2\right] \\
&= \mathbb{E}_{T \subseteq_p [n]} \mathbb{E}_{z \in \{-1,1\}^{|\overline{T}|}} \left[\mathbb{1}_{S \subseteq T} \cdot \left(\sum_{S' \subseteq \overline{T}} \widehat{f}(S \cup S') \prod_{j \in S'} z_j\right)^2\right] \\
&= \mathbb{E}_{T \subseteq_p [n]} \left[\mathbb{1}_{S \subseteq T} \cdot \mathbb{E}_{z \in \{-1,1\}^{|\overline{T}|}} \left[\left(\sum_{S' \subseteq \overline{T}} \widehat{f}(S \cup S') \prod_{j \in S'} z_j\right)^2\right]\right] \\
&= \mathbb{E}_{T \subseteq_p [n]} \left[\mathbb{1}_{S \subseteq T} \cdot \left(\sum_{S' \subseteq \overline{T}} \widehat{f}(S \cup S')^2\right)\right] \\
&= \sum_{U \subseteq [n]} \widehat{f}(U)^2 \cdot \Pr_\rho[\rho^{-1}(*) \cap U = S].
\end{aligned}$$

The fourth equality follows from the Parseval's Theorem. □

Using the above fact, we now bound the Fourier weight at a level. Recall, for $d \in [n]$, $W^d[f] := \sum_{S:\, |S|=d} \widehat{f}(S)^2$.

Proposition 4. *Let* $f : \{-1,1\}^n \to \mathbb{R}$*,* $d \in [n]$*, and* $p \in [0,1]$*. Then,*

$$\mathbb{E}_{\rho \in R_p}\left[W^d[f|_\rho]\right] = \sum_{k=d}^{n} W^k[f] \cdot \Pr[Bin(k,p) = d],$$

where $Bin(k,p)$ *denotes the number of successes given by the binomial distribution with* k *trials and success probability being* p*.*

Proof.

$$
\begin{aligned}
\mathbb{E}_{\rho\in R_p}\left[W^d[f|_\rho]\right] &= \mathbb{E}_{\rho\in R_p}\left[\sum_{S\colon |S|=d}\widehat{f|_\rho}(S)^2\right]\\
&= \sum_{S\colon |S|=d}\mathbb{E}_{\rho\in R_p}\left[\widehat{f|_\rho}(S)^2\right]\\
&= \sum_{S\colon |S|=d}\sum_{U\subseteq[n]}\widehat{f}(U)^2\cdot\Pr_\rho[\rho^{-1}(*)\cap U=S]\\
&= \sum_{U\subseteq[n]}\widehat{f}(U)^2\cdot\sum_{S\colon |S|=d}\Pr_\rho[\rho^{-1}(*)\cap U=S]\\
&= \sum_{k=d}^{n}\sum_{U\colon |U|=k}\widehat{f}(U)^2\sum_{S\colon |S|=d}\Pr_\rho[\rho^{-1}(*)\cap U=S]\\
&= \sum_{k=d}^{n}\sum_{U\colon |U|=k}\widehat{f}(U)^2\Pr[Bin(k,p)=d]\\
&= \sum_{k=d}^{n}W^k[f]\cdot\Pr[Bin(k,p)=d].
\end{aligned}
$$

The third equality followed from Proposition 3. $\square$

Using the above proposition we can now relate the Fourier tail of f with the Fourier tail of restrictions.

Lemma 5. *For any $f\colon\{-1,1\}^n\to\mathbb{R}$, $k\geq 0$, and $p\in[0,1]$, we have*

$$W^{\geq k}[f]\leq 2\cdot\mathbb{E}_{\rho\in R_p}\left[W^{\geq\lfloor kp\rfloor}[f|_\rho]\right],$$

where R_p is the set of p-random restrictions.

Proof.

$$
\begin{aligned}
\mathbb{E}_{\rho\in R_p}\left[W^{\geq\lfloor kp\rfloor}[f|_\rho]\right] &= \sum_{\ell\geq\lfloor kp\rfloor}W^\ell[f]\cdot\Pr[Bin(\ell,p)\geq\lfloor kp\rfloor]\\
&\geq \sum_{\ell\geq k}W^\ell[f]\cdot\Pr[Bin(\ell,p)\geq\lfloor kp\rfloor]\\
&\geq \sum_{\ell\geq k}W^\ell[f]\cdot\frac{1}{2}\\
&= \frac{W^{\geq k}[f]}{2}.
\end{aligned}
$$

The first equality follows from the Proposition 4, and the third inequality follows because median of the Binomial distribution $Bin(\ell,p)\geq\lfloor\ell p\rfloor\geq\lfloor kp\rfloor$. $\square$

We now complete the proof of Theorem 1 assuming the following lemma that says with high probability the degree of the restricted function is low.

Lemma 6. *Let $f\colon \{0,1\}^n \to \{0,1\}$ be computable by a circuit of size s and depth d. Then for all $t > 0$,*

$$\Pr_{\rho}[\mathsf{deg}(f|_\rho) > t] \leq s \cdot 2^{-t},$$

where ρ is a $\frac{1}{10^d t^{d-1}}$-random restriction.

Proof of Theorem 1. From Lemma 5, we have

$$\begin{aligned}
W^{\geq k}[f] &\leq 2 \cdot \mathbb{E}_{\rho\in R_p}\left[W^{\geq \lfloor kp \rfloor}[f|_\rho]\right] \\
&\leq 2 \cdot \mathbb{E}_{\rho\in R_p}\left[\mathbb{1}_{\mathsf{deg}(f|_\rho)\geq \lfloor kp \rfloor}\right] \\
&= 2 \cdot \Pr_{\rho}\left[\mathsf{deg}(f|_\rho) \geq \lfloor kp \rfloor\right] \\
&\leq 2 \cdot s \cdot 2^{-k^{1/d}/10}.
\end{aligned}$$

The second inequality follows because $0 \leq W^{\geq \lfloor kp \rfloor}[f|_\rho] \leq 1$, and the last inequality follows from choosing $p = 1/10^d t^{d-1}$ where $t = \lfloor kp \rfloor$ and using Lemma 6. From which it follows that $p = \frac{1}{10k^{\frac{d-1}{d}}}$, and thereby $t = k^{1/d}/10$. ☐

We now prove the main lemma, Lemma 6. We will use the following two versions of Håstad's switching lemma in the proof.

Theorem 7 (Switching lemma version 2)**.** *Let f be k-DNF and ρ be a p-random restriction where $p \in [0,1]$. Then, for all $t > 0$,*

$$\Pr_{\rho}[\mathsf{cnf\text{-}width}(f|_\rho) \geq t] \leq (5pk)^t.$$

Similarly one has an analogous version where we hit a k-CNF with a random restriction and bound the $\mathsf{dnf\text{-}width}$ of the restriction. We note another version where the degree of the restricted function is bounded.

Theorem 8 (Switching lemma version 3)**.** *Let f be k-DNF (or, k-CNF) and ρ be a p-random restriction where $p \in [0,1]$. Then, for all $t > 0$,*

$$\Pr_{\rho}[\mathsf{deg}(f|_\rho) \geq t] \leq (5pk)^t.$$

Proof of Lemma 6. The proof is similar to the proof of lower bound for parity we did in the last class. Again we restrict the circuit in stages. We also assume wlog that the circuit is alternating.

Stage 0: We restrict the circuit with $\frac{1}{10}$-random restrictions to ensure that the fan-in of the gates at the bottom level is at most t with very high probability.

Suppose the fan-in of a gate is more than $2t$. Then, the probability that this gate is not removed by the random restriction is at most $(1.1/2)^{2t} < 2^{-t}$. In the other case when the fan-in is at most $2t$. The probability that this gate has fan-in more than t is upper bounded by the probability that at least t inputs get assigned $*$. This is bounded by at most $\binom{2t}{t}(0.1)^t < 2^{-t}$. Therefore, the probability that some gate at the bottom has fan-in more than t is at most $s_1 \cdot 2^{-t}$, where s_1 is the number of gates at the bottom level.

Stage 1: To the circuit obtained after Stage 0, we apply $\frac{1}{10t}$-random restrictions, and use Theorem 7 to switch the gates at level 2 with t-CNFs. The probability of failure at this stage is at most $s_2 \cdot (5 \cdot \frac{1}{10t} \cdot t)^t \leq s_2 \cdot 2^{-t}$. After the switching we obtain a circuit of depth $d-1$.

We recursively repeat Stage 1 $(d-2)$-times to obtain a depth-2 circuit with bottom fan-in at most t. For the last time, we now again apply $\frac{1}{10t}$-random restrictions and use Theorem 8 to obtain a restricted function with degree at most t with high probability.

By union bound the total failure probability is bounded by $s \cdot 2^{-t}$. Thus, we obtain the lemma.

$\square$